TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, December 1970

GROUP ALGEBRA MODULES. III

BY
S. L. GULICK(1), T.-S. LIU(1) AND A. C. M. VAN ROOIJ

Abstract. Let Γ be a locally compact group and K a Banach space. The left $L^1(\Gamma)$ module K is by definition absolutely continuous under the composition $*$ if for $k \in K$ there exist $f \in L^1(\Gamma)$, $k' \in K$ with $k = f * k'$. If the locally compact Hausdorff space X is a transformation group over Γ and has a measure quasi-invariant with respect to Γ, then $L^1(X)$ is an absolutely continuous $L^1(\Gamma)$ module—the main object we study. If $Y \subseteq X$ is measurable, let L_Y consist of all functions in $L^1(X)$ vanishing outside Y. For $\Omega \subseteq \Gamma$ not locally null and B a closed linear subspace of K, we observe the connection between the closed linear span (denoted $L_\Omega * B$) of the elements $f * k$, with $f \in L_\Omega$ and $k \in B$, and the collection of functions of B shifted by elements in Ω. As a result, a closed linear subspace of $L^1(X)$ is an L_Z for some measurable $Z \subseteq X$ if and only if it is closed under pointwise multiplication by elements of $L^\infty(X)$. This allows the theorem stating that if $\Omega \subseteq \Gamma$ and $Y \subseteq X$ are both measurable, then there is a measurable subset Z of X such that $L_\Omega * L_Y = L_Z$. Under certain restrictions on Γ, we show that this Z is essentially open in the (usually stronger) orbit topology on X. Finally we prove that if Ω and Y are both relatively sigma-compact, and if also $L_\Omega * L_Y \subseteq L_Y$, then there exist Ω_1 and Y_1 locally almost everywhere equal to Ω and Y respectively, such that $\Omega_1 Y_1 \subseteq Y_1$; in addition we characterize those Ω and Y for which $L_\Omega * L_\Omega = L_\Omega$ and $L_\Omega * L_Y = L_Y$.

1. **Introduction.** This paper, and the one which follows, arise quite naturally from our earlier papers [3] and [4]. Let us see how. Take Γ as a locally compact group, and $L^1(\Gamma)$ the Banach space of integrable functions on Γ. If we let K be an arbitrary left $L^1(\Gamma)$ module, we may inquire what are the left module homomorphisms from $L^1(\Gamma)$ to K. In [3], amongst other things, we give a (not quite complete) solution to the general question, and then give complete solutions in case $K = L^p(\Gamma)$, $p \in [1, \infty]$. In [4] we assume that Γ acts on a given locally compact space X as a transformation group and that m_X is a measure on X quasi-invariant with respect to Γ. Then we show that $L^p(X)$ may be rendered as a left $L^1(\Gamma)$ module, to which we may ask what are the left module homomorphisms from $L^1(\Gamma)$ to $L^p(X)$.

The present investigations start at that point. In this paper we discuss the more general aspects of Banach spaces K which can be represented as left $L^1(\Gamma)$ modules. We denote the module composition by $*$. We pay particular attention to those modules whose elements are factorable (i.e., $k \in K$ implies that there is an

Received by the editors June 17, 1969.

AMS 1969 *subject classifications*. Primary 4680, 2220.

Key words and phrases. Transformation group, quasi-invariant measure, absolutely continuous measure, approximate identity, factorable, orbit topology, vanishing algebra, group algebra module.

(1) Research supported in part by the National Science Foundation.

$f \in L^1(\Gamma)$ and $k' \in K$ such that $k = f * k'$). Such spaces we call absolutely continuous modules. For each element in such a K we can describe the notion of left shift by elements of Γ, and each such shift by $\sigma \in \Gamma$ is continuous as a function of σ.

A major reason for our study of absolutely continuous modules appears in §3. For $\Omega \subseteq \Gamma$, let L_Ω consist of all $L^1(\Gamma)$ functions vanishing off Ω. If B is a closed subspace of K, we denote by $L_\Omega * B$ the closed linear span of elements of the form $f * k$, $f \in L_\Omega$ and $k \in K$. The fundamental Decomposition Theorem 3.2 states that if Ω is not locally null, then any such $L_\Omega * B$ can be approximated by sums of shifts of B by elements essentially (to be made precise in the text) in Ω. If Y is measurable in X, and L_Y has a meaning analogous to L_Ω, then $L_\Omega * L_Y$ may be approximated by sums of shifts of L_Y. In particular, $L_\Gamma * L_\Omega = L_\Gamma$, the whole space! Together with Theorem 3.9, which determines that a closed linear subspace of $L^1(X)$ is an L_Z for some measurable $Z \subseteq X$ if and only if it is closed under pointwise multiplication by $L^\infty(X)$, these results allow us to prove in Theorem 3.10 that if $\Omega \subseteq \Gamma$ and $Y \subseteq X$ are both measurable, then there exists a measurable subset Z of X such that $L_\Omega * L_Y = L_Z$. We terminate the section with a sort of counterpart to the decomposition and a corollary of value in later studies.

In the rest of this paper we analyze the set Z occurring above—under the stipulation that if Γ_0 is a sigma-compact open subgroup of Γ, then $m_X(\Gamma_0 x) > 0$ for all $x \in X$. This condition gives us certain continuity conditions on the convolution which we utilize, and without the condition Z is unmanageable. It turns out that Z is essentially open—in a certain natural topology which ordinarily is stronger than the given topology on X. This new topology we call the orbit topology since it is described in terms of the orbits Γx, $x \in X$, and we discuss it in §4 when we are in the process of determining Z.

In §5 we study the relationship between the two notions $\Omega Y \subseteq Y$ and $L_\Omega * L_Y \subseteq L_Y$. That $\Omega Y \subseteq Y$ implies $L_\Omega * L_Y \subseteq L_Y$ is true and easy to prove. The converse would say that if $L_\Omega * L_Y \subseteq L_Y$ then there exist Ω_1, Y_1 l.a.e. equal to Ω, Y respectively, such that $\Omega_1 Y_1 \subseteq Y_1$. It is as yet not known, even if $X = \Gamma$ and $Y = \Omega$, except when Ω is relatively sigma-compact [6]. Nevertheless, we prove it when both Ω and Y are relatively sigma-compact. We conclude the paper by characterizing those Ω and Y for which $L_\Omega * L_\Omega = L_\Omega$ and $L_\Omega * L_Y = L_Y$.

2. **Setting.** We begin the definitions and notations by prescribing $\varnothing$ to be the empty set. If A and B are two sets, then $B \backslash A$ is the complement of $A \cap B$ in B. Let Γ be a locally compact group with identity 1, m a left Haar measure on Γ, and L_Γ the Banach space of integrable functions on Γ, with the usual norm $\| \ \|_1$. For $\sigma \in \Gamma$ and $f \in L_\Gamma$ we have the left shift f_σ (also in L_Γ), defined by $f_\sigma(\tau) = f(\sigma\tau)$, $\tau \in \Gamma$, and the right shift f^σ (in L_Γ), defined by $f^\sigma(\tau) = f(\tau\sigma)\Delta(\sigma)$, $\tau \in \Gamma$, where Δ is the modular function for Γ. For $f \in L_\Gamma$ we let $f' \in L_\Gamma$ be defined by $f'(\tau) = \Delta(\tau^{-1}) f(\tau^{-1})$. If $\Omega \subseteq \Gamma$, we let L_Ω denote the collection of functions in L_Γ which vanish almost everywhere in $\Gamma \backslash \Omega$.

Let X be a locally compact (Hausdorff) space. We say that Γ acts as a transformation group on X if Γ is a group of homeomorphisms on X such that the map $\Gamma \times X \to X$ defined by $(\sigma, x) \to \sigma x$, $\sigma \in \Gamma$, $x \in X$, is jointly continuous. If m_X is a positive Radon measure on X with the property that if $Y \subseteq X$ and $m_X(Y)=0$, then $m_X(\sigma Y)=0$ for all $\sigma \in \Gamma$, then we say that m_X is quasi-invariant. The functions integrable on X with respect to m_X describe the space L_X, under the usual L^1 norm. For $Y \subseteq X$ and $p \in [1, \infty]$, $L_Y^p = \{f \in L^p(X) : f=0 \text{ l.a.e. on } X \backslash Y\}$. The characteristic function of $Y \subseteq X$ is written ξ_Y. We abbreviate "almost everywhere" to "a.e.", and "locally almost everywhere" to "l.a.e." We use the notation $Y \subseteq Z$ l.a.e. to mean that $Z \backslash Y$ is locally null. Then $Y=Z$ l.a.e. means that $Y \subseteq Z$ l.a.e. and $Z \subseteq Y$ l.a.e. For measure-theoretic notations we generally follow [5]. Let L_X^∞ denote the measurable, essentially bounded functions on X. We let I denote an indexing set (to serve the purpose). We denote by $\boldsymbol{R}$ the additive group of real numbers with the usual topology.

Throughout the paper K will denote a Banach space, K^* the topological conjugate (dual) space under its usual dual norm. For every $\mu \in M(\Gamma)$ and every bounded continuous map $F: \Gamma \to K$ there exists by Proposition 8 of §1 of [1] a unique element $\int F\, d\mu \in K$ such that

$$k^*\left(\int F\, d\mu\right) = \int_X (k^* \circ F)\, d\mu, \qquad k^* \in K^*.$$

Further, if K_1 is another Banach space and $T: K \to K_1$ is a continuous linear map, then $\int (T \circ F)\, d\mu = T(\int F\, d\mu)$.

If $f \in L_\Gamma$, there is a unique $\mu \in M(\Gamma)$ with $d\mu(\sigma) = f(\sigma)\, d\sigma$, $d\sigma$ representing the element of the left invariant Haar measure. Instead of $\int F\, d\mu$ we shall write $\int f(\sigma) F(\sigma)\, d\sigma$. Then $\|\int f(\sigma)F(\sigma)\, d\sigma\| \leqq \int_\Gamma |f(\sigma)| \cdot \|F(\sigma)\|\, d\sigma$.

Let K be a Banach space. We call $(K, *)$ a left L_Γ-module if K is a Banach space over the same scalar field as L_Γ and if $*$ is a bilinear operation with the following properties:

(α) $*: L_\Gamma \times K \to K$,

(β) $(f * g) * k = f * (g * k)$, $f, g \in L_\Gamma$, $k \in K$,

(γ) $\|f * k\| \leqq \|f\|_1 \|k\|$, $f, g \in L_\Gamma$, $k \in K$.

We should not confuse the module composition with convolution, even though they are given by the same symbol. The former acts on $L_\Gamma \times K$, and the latter on $L_\Gamma \times L_\Gamma$. A simple glance to the right and left of $*$ should tell which space $*$ acts on.

For any L_Γ-module K let us denote by K_{abs} the space $\{f * k : f \in L_\Gamma, k \in K\}$. Then K_{abs} is a closed submodule of K (Corollary 2.3 of [4]), and $(K_{\mathrm{abs}})_{\mathrm{abs}} = K_{\mathrm{abs}}$. (This follows from the fact that L_Γ is factorable; see [2].) If $K = K_{\mathrm{abs}}$, we say that K is *absolutely continuous*. It is easy to see that if K is absolutely continuous, then for an approximate identity $(u_\iota)_{\iota \in I}$ in L_Γ, we have $\lim_\iota (u_\iota * k) = k$ for each $k \in K$. As K_{abs} is closed, the converse is also true.

If K is absolutely continuous, then every element $\sigma \in \Gamma$ defines a unique linear isometry $k \to k_\sigma$ of K onto K such that

$$f * k = \int f(\tau) k_{\tau^{-1}} \, d\tau,$$
$$(f * k)_\sigma = f_\sigma * k, \quad \text{and} \quad f * k_\sigma = f^\sigma * k \qquad (f \in L_\Gamma, k \in K, \sigma \in \Gamma).$$

k_σ is called the shift of k by σ. It is jointly continuous and satisfies $(k_\sigma)_\tau = k_{\sigma\tau}$, $k_1 = k$. We shall omit the proof of these facts. We only mention that k_σ may be defined by $k_\sigma = \lim_i u_i^\sigma * k$ $(k \in K)$.

We defend the terminology "absolutely continuous module" by noting that if $K = M(\Gamma)$, then $\mu \in M(\Gamma)$ is absolutely continuous in the conventional sense (with respect to left Haar measure) if and only if $\mu \in M(\Gamma)_{\text{abs}}$.

We describe other examples of L_Γ-modules, some of which we will find are absolutely continuous, and others not. To begin with, let Γ be a locally compact transformation group acting on a locally compact Hausdorff space X, as described above. We make $C_\infty(X)$ and $M(X)$ into L_Γ-modules by

$$f * k(x) = \int_\Gamma f(\sigma) k(\sigma^{-1}x) \, d\sigma \qquad (f \in L_\Gamma, k \in C_\infty(X), x \in X)$$
$$f * \mu(k) = \int_\Gamma (f' * k) \, d\mu \qquad (f \in L_\Gamma, \mu \in M(X), k \in C_\infty(X)).$$

$C_\infty(X)$ is absolutely continuous ([4, 4.11]) and the shift is given by

$$k_\sigma(x) = k(\sigma x) \qquad (\sigma \in \Gamma, x \in X, k \in C_\infty(X)).$$

In general, $M(X)$ will not be absolutely continuous. The space $M(X)_{\text{abs}}$ has been discussed in [7].

Let Γ act on X as above, and let m_X be a quasi-invariant measure on X. The realization of L_X as the space of all elements of $M(X)$ that are absolutely continuous with respect to m_X makes L_X an absolutely continuous submodule of $M(X)$ (Theorem 4.11 of [4]). In [4, §4] the authors have constructed a positive measurable function J on $\Gamma \times X$ such that for $f \in L_\Gamma$ and $k \in L_X$,

$$f * k(x) = \int_\Gamma f(\sigma) k(\sigma^{-1}x) J(\sigma^{-1}, x) \, d\sigma$$

for locally almost all $x \in X$. By means of this J we can make every L_X^p $(1 \leqq p \leqq \infty)$ into an L_Γ-module by defining

$$f * k(x) = \int_\Gamma f(\sigma) k(\sigma^{-1}x) J(\sigma^{-1}, x)^{p^{-1}} \, d\sigma \quad (\text{l.a.e. } x \in X)$$

for all $f \in L_\Gamma$, $k \in L_X^p$. As was shown in [4, Theorem 4.11] L_X^p is absolutely continuous if $p < \infty$. Generally L_X^∞ is not. The canonical map $C_\infty(X) \to L_X^\infty$ is a module homomorphism. In particular, if supp $m_X = X$, $C_\infty(X)$ is an absolutely continuous submodule of L_X^∞.

One final note on concrete examples of L_Γ absolutely continuous modules. There exist examples for which the composition is not a generalized convolution. Take X to be an abelian locally compact group and let Γ be the character group of X. For $f \in L_\Gamma$ and $k \in L_X$, put $f * k = \hat{f}k$.

Absolutely continuous L_Γ-modules have some inheritance properties. In the first place, we have already mentioned that closed submodules of absolutely continuous L_Γ-modules are themselves absolutely continuous. Next we come to sums and intersections of absolutely continuous L_Γ-modules, which we define forthwith. Let K_1 and K_2 be L_Γ-modules and let H be a closed subspace of the product $K_1 \times K_2$. We require that $(f * k_1, f * k_2) \in H$ for any $f \in L_\Gamma$, $(k_1, k_2) \in H$. The vector spaces H and $(K_1 \times K_2)/H$ are turned into Banach spaces $K_1 \wedge_H K_2$ and $K_1 \vee_H K_2$, respectively, by the definitions

$$\|(k_1, k_2)\| = \max(\|k_1\|, \|k_2\|),$$
$$\|(k_1, k_2)+H\| = \inf\{\|k_1'\| + \|k_2'\| : k_1' \in K_1, k_2' \in K_2 \text{ and } (k_1', k_2') \equiv (k_1, k_2)+H\}.$$

($K_1 \wedge_H K_2$ is called the intersection of K_1 and K_2; $K_1 \vee_H K_2$ is their sum.)

The proofs of these facts and a general investigation of these Banach spaces occur in [8]. Carrying on, we render $K_1 \wedge_H K_2$ and $K_1 \vee_H K_2$ as L_Γ-modules by the quite natural formulas

$$f * (k_1, k_2) = (f * k_1, f * k_2), \qquad f \in L_\Gamma, (k_1, k_2) \in H,$$
$$f * [(k_1, k_2)+H] = (f * k_1, f * k_2)+H, \qquad f \in L_\Gamma, k_1 \in K_1, k_2 \in K_2.$$

Furthermore, if K_1 and K_2 are absolutely continuous, then so are $K_1 \wedge_H K_2$ and $K_1 \vee_H K_2$. In fact, it is simple to compute that

$$(K_1 \wedge_H K_2)_{\mathrm{abs}} = H \cap \{(k_1, k_2) : k_1 \in (K_1)_{\mathrm{abs}}, k_2 \in (K_2)_{\mathrm{abs}}\},$$
$$(K_1 \vee_H K_2)_{\mathrm{abs}} = \{(k_1, k_2)+H : k_1 \in (K_1)_{\mathrm{abs}}, k_2 \in (K_2)_{\mathrm{abs}}\}.$$

If we wish to investigate the dual K^* of an L_Γ-module K, a natural composition is defined by

$$(f * k^*)k = k^*(f' * k), \qquad f \in L_\Gamma, k \in K, k^* \in K^*,$$

and endowed with it, K^* is an L_Γ-module. We have already used this formula to define a module structure on $M(X) = C_\infty(X)^*$. In general, K^* will not be absolutely continuous, even if K is. For example, take $K = L_X$ where $X = \Gamma = \boldsymbol{R}$. Then K is absolutely continuous, while $K^* = L^\infty(X)$ is not. On the other hand, if we use the criterion for absolute continuity of K that K must be factorable, then an application of the Hahn-Banach theorem shows us that if K is not absolutely continuous, then under no circumstance can K^* be. In fact K is absolutely continuous if and only if K^* is order-free. (An L_Γ-module K is said to be order-free if for each $k \in K$, $f * k = 0$ for all $f \in L_\Gamma$ implies $k = 0$.) For a corollary we observe that all reflexive order-free modules are absolutely continuous and have absolutely continuous duals.

3. **A decomposition theorem and its consequences.** Before we can give the decomposition theorem in the form we desire, we must have a preliminary discussion. Assume that X is locally compact and Hausdorff and has a positive Radon measure m_X. For a measurable set Y we define the two operators i and d as follows:

$$iY = \{x \in X : \text{there exists a measurable neighborhood } V \text{ of } x \text{ such that } m_X(V \setminus Y) = 0\},$$

$$dY = \{x \in X : \text{for every measurable neighborhood } V \text{ of } x,\ m_X(V \cap Y) > 0\}.$$

The operators i and d are not new; they have been discussed in [6]. Their more elementary properties are:

$Y^0 \subseteq iY = (iY)^0 \subseteq dY = (\mathrm{Cl}\,(dY)) \subseteq \overline{Y}$, $iY \subseteq Y$ l.a.e., and $Y \subseteq dY$ l.a.e.

$\sigma(iY) = i(\sigma Y)$ and $\sigma(dY) = d(\sigma Y)$ for every $\sigma \in \Gamma$.

$Y \subseteq Y'$ l.a.e. implies $iY \subseteq IY'$ and $dY \subseteq dY'$.

Verbally, iY is an open set containing the interior of Y, while dY is a closed set contained in the closure of Y.

In particular, the operators d and i are defined in Γ itself. One of the basic properties of d is the following.

3.1. LEMMA. *Let $\Omega \subseteq \Gamma$ be measurable, $\gamma \in d\Omega$. Then L_Γ contains an approximate identity $(u_i)_{i \in I}$ such that $\|u_i\|_1 = 1$ and $(u_i)^{\gamma^{-1}} \in L_\Omega$ for every i.*

Proof. For I we take the net of all compact neighborhoods of $1 \in \Gamma$, made into a directed set by the definition $\Phi_1 \prec \Phi_2$, if $\Phi_1 \supseteq \Phi_2$. For $\Phi \in I$ let

$$u_\Phi = [m(\Phi \cap \Omega\gamma^{-1})]^{-1} \xi_{\Phi \cap \Omega\gamma^{-1}}.$$

(Note that $m(\Phi \cap \Omega\gamma^{-1}) \neq 0$ because $\gamma \in d\Omega$.) Then $(u_\Phi)^{\gamma^{-1}} \in L_\Omega$ and $\|u_\Phi\|_1 = 1$. Take $f \in L_\Gamma$, $\varepsilon > 0$. The set $\Phi_1 = \{\sigma \in \Gamma : \|f_{\sigma^{-1}} - f\| < \varepsilon\}$ is a neighborhood of $1 \in \Gamma$. It is now easy to see that $\|(u_\Phi * f) - f\|_1 < \varepsilon$ for all $\Phi \in I$ such that $\Phi \subseteq \Phi_1$. In fact, for such Φ,

$$\begin{aligned} \|(u_\Phi * f) - f\|_1 &= \left\| \int_\Gamma u_\Phi(\sigma) f_{\sigma^{-1}}\, d\sigma - \int_\Gamma u_\Phi(\sigma) f\, d\sigma \right\| \\ &\leqq \int_\Gamma u_\Phi(\sigma) \|f_{\sigma^{-1}} - f\|\, d\sigma < \varepsilon. \end{aligned}$$

Let B be a closed linear subspace of an absolutely continuous L_Γ-module K. For $\sigma \in \Gamma$ we denote $\{k_\sigma : k \in B\}$ by B_σ. For $\Omega \subseteq \Gamma$ measurable we indicate by $L_\Omega * B$ the closed linear subspace of K generated by $\{f * k : f \in L_\Omega, k \in B\}$. If $\Gamma = X$ and $B \subseteq L_X$ let $B' = \{f' \in L_X : f \in B\}$. In particular, if $Y \subseteq X = \Gamma$, then $L_{Y^{-1}} * L_{\Omega^{-1}} = (L_Y)' * (L_\Omega)' = (L_\Omega * L_Y)'$ by direct computation.

Now we are ready for the decomposition theorem.

3.2. MODULE DECOMPOSITION THEOREM. *Let K be an absolutely continuous module over L_Γ. For any measurable $\Omega \subseteq \Gamma$ and any closed subspace B of K we have*

$$L_\Omega * B = \mathrm{Cl}\left(\sum_{\sigma \in d\Omega} B_{\sigma^{-1}}\right).$$

Proof. Let $\sigma \in d\Omega$ and $k \in B$. By Lemma 3.1 there is an approximate identity $(u_i)_{i\in I}$ in L_Γ such that $(u_i)^{\sigma^{-1}} \in L_\Omega$ for every i. Then $k_{\sigma^{-1}} = \lim_i ((u_i)^{\sigma^{-1}} * k) \in L_\Omega * B$. Thus $\mathrm{Cl}\,(\sum_{\sigma\in d\Omega} B_{\sigma^{-1}}) \subseteq L_\Omega * B$. Conversely, let $f \in L_\Omega$, $k \in B$. For locally almost all $\tau \in \Omega$, we have $\tau \in d\Omega$, so that for these τ, $k_{\tau^{-1}} \in \mathrm{Cl}\,(\sum_{\sigma\in d\Omega} B_{\sigma^{-1}})$.In other words, $k \in B$ implies $k_{\tau^{-1}} \in \mathrm{Cl}\,(\sum_{\sigma\in d\Omega} B_{\sigma^{-1}})$ for locally almost every $\tau \in \Omega$. Take any $k^* \in K^*$ such that $k^* = 0$ on $\sum_{\sigma\in d\Omega} B_{\sigma^{-1}}$. Then $k^*(f * k) = \int_\Gamma f(\sigma) k^*(k_{\sigma^{-1}})\, d\sigma = 0$ for all $f \in L_\Omega$, $k \in B$, with the result that $k^* = 0$ on $L_\Omega * B$. Thus $L_\Omega * B \subseteq \mathrm{Cl}\,(\sum_{\sigma\in d\Omega} B_{\sigma^{-1}})$.

We mention that without the hypothesis of absolute continuity on K the conclusion may be invalid. Take, for example, $\Gamma = X = \boldsymbol{R}$. Let $B = K = L_X^\infty$. Then for each $\sigma \in \Gamma$, $B_{\sigma^{-1}} = L_X^\infty$, while $L_\Omega * B$ is the collection of uniformly continuous functions on X.

Several consequences follow directly.

3.3. Corollary. $L_\Omega * B = L_{d\Omega} * B$.

Corollaries 3.4 and 3.5 concern the case where Γ is a transformation group acting on X and X is endowed with a quasi-invariant measure. For $Y \subseteq X$ measurable and $\sigma \in \Gamma$ we have $(L_Y)_{\sigma^{-1}} = L_{\sigma Y}$. Thus

3.4. Corollary. *If $Y \subseteq X$ is measurable, then* $L_\Omega * L_Y = \mathrm{Cl}\,(\sum_{\sigma\in d\Omega} L_{\sigma Y})$.

3.5. Corollary. *Let Y and Z be measurable subsets of X and let $L_\Omega * L_Y \subseteq L_Z$. Then*

(i) *For every $\sigma \in d\Omega$, $\sigma Y \subseteq Z$ l.a.e.*

(ii) *$d\Omega\, dY \subseteq dZ$, so that if L_Ω is a subalgebra of L_Γ, then $d\Omega$ is a subsemigroup of Γ.*

(iii) *$d\Omega i Y \subseteq iZ$, so that if L_Ω is a subalgebra of L_Γ, then $i\Omega$ is a subsemigroup of Γ.*

If, in addition, $X = \Gamma$, then also

(iv) *For every $\sigma \in dY$, $\Omega\sigma \subseteq Z$ l.a.e.*

(v) *$i\Omega\, dY \subseteq iZ$.*

Proof. By the preceding corollary, if $\sigma \in d\Omega$, then $L_{\sigma Y} \subseteq L_Z$, so that $\sigma Y \subseteq Z$ l.a.e., proving (i). Then $\sigma(dY) = d(\sigma Y) \subseteq dZ$, and $\sigma(iY) = i(\sigma Y) \subseteq iZ$, thus proving both (ii) and (iii). Parts (iv) and (v) follow from (i) and (ii) via the formulas

$$L_{Y^{-1}} * L_{\Omega^{-1}} = (L_Y)' * (L_\Omega)' = (L_\Omega * L_Y)' \subseteq (L_Z)' = L_{Z^{-1}}.$$

For later use we file away yet another consequence.

3.6. Corollary. *If $\Omega \subseteq \Gamma$ is measurable and not locally null, then* $L_\Gamma * L_\Omega = L_\Gamma$.

Proof. Since Ω is not locally null, $d\Omega \neq \varnothing$, so let $\tau \in d\Omega$. Then

$$(L_\Gamma * L_\Omega)' = (L_\Omega)' * (L_\Gamma)' = L_{\Omega^{-1}} * L_\Gamma = \mathrm{Cl}\left(\sum_{\sigma\in d\Omega} L_{\sigma^{-1}\Gamma}\right) \supseteq L_{\tau^{-1}\Gamma} = L_\Gamma.$$

The space B employed in the last two corollaries has been contained in L_X. For a moment let us switch our attention to C_Y, the collection of all functions $k \in C_\infty(X)$ which vanish outside Y. Then we have

3.7. COROLLARY. $L_\Omega * C_Y = \mathrm{Cl}\,(\sum_{\sigma\in d\Omega} C_{\sigma Y}) = L_{d\Omega} * C_Y$.

3.8. COROLLARY. *If Y is open in X, then* $L_\Omega * C_Y = C_{(d\Omega)Y}$.

Proof. Since evidently $\mathrm{Cl}\,(\sum_{\sigma\in d\Omega} C_{\sigma Y}) \subseteq C_{(d\Omega)Y}$, we need only prove the opposite inclusion. To this end, let $f \in C_{(d\Omega)Y}$ with compact support. Since Y is open and $\mathrm{supp} f$ is compact, there exist $\tau_1, \ldots, \tau_n \in d\Omega$ such that $\mathrm{supp} f \subseteq \bigcup_{i=1}^n \tau_i Y$. By using a partition of unity one can construct $f_1, \ldots, f_n \in C(X)$ with $\sum_{i=1}^n f_i = f$, such that each f_i has compact support contained in $\tau_i Y$. Then $f \in \sum_{i=1}^n C_{\tau_i Y} \subseteq \sum_{\sigma\in d\Omega} C_{\sigma Y}$.

Because of Corollary 3.8, we quite involuntarily might conjecture that at least when Y is open in X, then $L_\Omega * L_Y = L_{(d\Omega)Y}$. In fact this is true, but the proof is by no means trivial. First we show that $L_\Omega * L_Y$ is an L_Z for an appropriate Z. For completeness we prove the following theorem.

3.9. THEOREM. *Let m_X be a positive Radon measure on a locally compact space X. Let B be a closed linear subspace of L_X. Then the following conditions are equivalent:*

(a) *There is a measurable set $Z \subseteq X$ such that $B = L_Z$.*

(b) *For all $k \in B$ and $j \in L_X^\infty$, $kj \in B$ (i.e., B is a module over $L_\infty(X)$ under pointwise multiplication).*

Proof. The implication (a) to (b) is evident. Now assume (b). By Theorem 11.39 of [5] there exists a family $\mathscr{F}$ of disjoint compact subsets of X such that for every U which is open in X and has finite measure, $\{F \in \mathscr{F} : m_X(U \cap F) > 0\}$ is countable, and such that $X \setminus \bigcup \mathscr{F}$ is locally null. It follows that a set $Y \subseteq X$ is measurable if and only if $Y \cap F$ is measurable for every $F \in \mathscr{F}$. For each $F \in \mathscr{F}$ let $\mathscr{X}_F = \{Y \subseteq F : Y$ is measurable and $\xi_Y \in B\}$. Consequently,

(α) If $Y_1, Y_2, \ldots$ is a sequence in $\mathscr{X}_F$, then $\bigcup_{n=1}^\infty Y_n \in \mathscr{X}_F$.

(β) If a measurable set Y is contained in an element of $\mathscr{X}_F$, then $Y \in \mathscr{X}_F$.

By (α) for every F there exists a $Z_F \in \mathscr{X}_F$ such that $m_X(Z_F) = \sup\{m_X(Y) : Y \in \mathscr{X}_F\}$. Then $Z = \bigcup\{Z_F : F \in \mathscr{F}\}$ is measurable by the comments above, and is the subset of X we desire. Now we show that $B = L_Z$. Inasmuch as both B and L_Z are closed modules over L_X^∞ under pointwise multiplication, it suffices to show that $\{Y \subseteq X : \xi_Y \in B\} = \{Y \subseteq X : \xi_Y \in L_Z\}$. To show it, first let $\xi_Y \in L_Z$ and assume that $Y \subseteq Z$ everywhere. Then for every $F \in \mathscr{F}$, we have $Y \cap F \subseteq Z \cap F = Z_F$, so that by ($\beta$), $Y \cap F \in \mathscr{X}_F$. Thus $\xi_{Y\cap F} \in B$. Since Y is of finite measure, there can exist only countably many $F \in \mathscr{F}$ such that $m_X(Y \cap F) > 0$. Hence $\xi_Y = \sum\{\xi_{Y\cap F} : F \in \mathscr{F}\} \in B$. On the other hand, let $\xi_Y \in B$. For each $F \in \mathscr{F}$, $\xi_{Y\cap F} = \xi_Y \xi_F \in B$, so that $Y \cap F \in \mathscr{X}_F$. Then $(Y \cap F) \cup Z_F \in \mathscr{X}_F$ by (α). It follows that $m_X((Y \cap F) \cup Z_F) \leqq m_X(Z_F)$. Hence, $Y \cap F \subseteq Z_F$ a.e., which means that $Y \subseteq Z$ l.a.e., and $\xi_Y \in L_Z$.

In 3.10–3.13, Γ is again a group of homeomorphisms of a space X on which we have a quasi-invariant measure m_X.

3.10. THEOREM. *For any measurable $\Omega \subseteq \Gamma$ and $Y \subseteq X$ there exists a measurable set $Z \subseteq X$ such that* $L_\Omega * L_Y = L_Z$.

Proof. Since each $L_{\sigma Y}$ is a module over L_X^∞, for $\sigma \in d\Omega$, this means that $\mathrm{Cl}\,(\sum_{\sigma \in d\Omega} L_{\sigma Y}) = L_\Omega * L_Y$ is also an L_X^∞ module.

We now arrive at the proposition promised following Corollary 3.8.

3.11. COROLLARY. *If $\Omega \subseteq \Gamma$ is measurable and $Y \subseteq X$ is open, then $L_\Omega * L_Y = L_{(d\Omega)Y}$.*

Proof. Let Z be as in the preceding theorem. Since Y is open, $Y \subset iY$. By Corollary 3.5(iii), $d\Omega\, Y \subset d\Omega i Y \subset iZ$. Since always $iZ \subseteq Z$ l.a.e., we have $(d\Omega)Y \subseteq Z$ l.a.e. On the other hand, $L_\Omega * L_Y \subseteq L_{(d\Omega)Y}$, so that $Z \subseteq (d\Omega)Y$ l.a.e. Consequently, $Z = (d\Omega)Y$ l.a.e., which is what we needed to prove.

With an added hypothesis we can go a step further.

3.12. THEOREM. *Let $\Omega \subseteq \Gamma$ be measurable and $Y \subseteq X$ open, and assume that $L_\Omega * L_Y \subseteq L_Y$. Then there exist an $\Omega' \subseteq \Gamma$ and an open $Y' \subseteq X$ such that $\Omega' = \Omega$ l.a.e. and $Y' = Y$ l.a.e., and $\Omega' Y' \subseteq Y'$.*

Proof. Let $\Omega' = \Omega \cap d\Omega$ and $Y' = iY$. Then $\Omega' = \Omega$ l.a.e. and $Y' = Y$ l.a.e. By the preceding corollary, $L_{(d\Omega)Y'} = L_\Omega * L_{Y'} \subseteq L_{Y'}$, so that $(d\Omega')Y' \subseteq Y'$ l.a.e. Note that since Y' is open, $(d\Omega')Y'$ is also open. Then $\Omega' Y' \subseteq (d\Omega')Y' \subseteq i\{(d\Omega')Y'\} \subseteq iY' = i(iY) = iY = Y'$.

There is a companion to this corollary—for Y closed in X—which we presently demonstrate.

3.13. THEOREM. *Let $\Omega \subseteq \Gamma$ be measurable and $Y \subseteq X$ closed, and assume that $L_\Omega * L_Y \subseteq L_Y$. Then there exist a set $\Omega' \subseteq \Gamma$ and a closed set $Y' \subseteq X$ such that $\Omega' = \Omega$ l.a.e. and $Y' = Y$ l.a.e., and $\Omega' Y' \subseteq Y'$.*

Proof. Let $\Omega' = \Omega \cap d\Omega$ and $Y' = dY$. By assumption, the Z of Theorem 3.10 has the property that $Z \subseteq Y$ l.a.e. Thus $dZ \subseteq dY$, whereupon $\Omega' Y' \subseteq d\Omega\, dY \subseteq dZ \subseteq dY = Y'$, by an application of Corollary 3.5(ii).

Theorem 3.10 says that if we are given measurable sets $\Omega \subseteq \Gamma$ and $Y \subseteq X$, then the collection of all $k \in L_X$ such that $k \in L_\Omega * L_Y$ can be represented as L_Z for an appropriately chosen $Z \subseteq X$; sometimes—at least when Y is open—we can describe Z in a simple form merely in terms of Ω and Y. Now let us turn the question around. Suppose we are given once again measurable sets $\Omega \subseteq \Gamma$ and $Y \subseteq X$, but this time we are interested in the collection of all $k \in L_X$ such that $L_\Omega * k \subseteq L_Y$. We will show that this collection forms an L_Z and we will describe Z in terms of Ω and Y.

First we have a preliminary proposition, a kind of counterpart to the Decomposition Theorem.

3.14. THEOREM. *Let K be an absolutely continuous module over L_Γ. Also let $\Omega \subseteq \Gamma$ be measurable and let B be a closed linear subspace of K. Then*

$$\{k \in K : L_\Omega * k \subseteq B\} = \bigcap_{\sigma \in d\Omega} B_\sigma.$$

Proof. Let $L_\Omega * k \subseteq B$ and $\sigma \in d\Omega$. Then, by Lemma 3.1, there is an approximate identity $(u_i)_{i\in I}$ in L_Γ with $(u_i)^{\sigma^{-1}} \in L_\Omega$ for every $i \in I$. This means that $k_{\sigma^{-1}} = \lim_i (u_i^{\sigma^{-1}} * k) \in B$, so that $k \in B_\sigma$, which therefore holds for all $\sigma \in d\Omega$. On the other hand, let $k \in \bigcap_{\sigma\in d\Omega} B_\sigma$ and let $k^* \in K^*$ with the property that $k^* = 0$ on B. Then $k^*(k_{\sigma^{-1}}) = 0$ for locally almost all $\sigma \in \Omega$. Thus, for any $f \in L_\Omega$, $k^*(f * k) = \int_\Gamma f(\sigma) k^*(k_{\sigma^{-1}})\, d\sigma = 0$. Since this is true for all $k^* \in K^*$ which vanish on B, we obtain $f * k \in B$. Consequently, $L_\Omega * k \subseteq B$.

We turn to more concrete examples.

3.15. Theorem. *Let $\Omega \subseteq \Gamma$ and $Y \subseteq X$ be measurable, $A = \{x \in X : \sigma x \in Y$ for locally almost all $\sigma \in \Omega\}$. Then A is measurable and $L_A^p = \{k \in L_X^p : L_\Omega * k \subseteq L_Y^p\}$ for every $p \in [1, \infty]$.*

Proof. First let $p = 1$. Inasmuch as $\bigcap_{\sigma\in d\Omega} (L_Y)_\sigma = \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}$ is a module over L_X^∞ under pointwise multiplication, the space $\{k \in L_X : L_\Omega * k \subseteq L_Y\}$ is of the form L_Z, by Theorem 3.9. All we need to prove, then, is that if D is compact in X, then $D \subseteq Z$ a.e. if and only if $D \subseteq A$ a.e. Therefore let $D \subseteq A$ a.e. This means that $\{\sigma \in \Omega : \sigma x \in X \setminus Y\}$ is locally null for almost all $x \in D$, so that for any compact subset Φ of Ω,

$$\begin{aligned} 0 &= \int_X \xi_D(x) \int_\Gamma \xi_\Phi(\sigma)\xi_{X\setminus Y}(\sigma x)\, d\sigma\, dx \\ &= \int_\Gamma \xi_\Phi(\sigma) \int_X \xi_D(x)\xi_{X\setminus Y}(\sigma x)\, dx\, d\sigma \\ &= \int_\Gamma \xi_\Phi(\sigma) m(D \cap \sigma^{-1}(X\setminus Y))\, d\sigma. \end{aligned}$$

Therefore $m_X(D \cap \sigma^{-1}(X\setminus Y)) = 0$ for locally almost all $\sigma \in \Omega$. By the quasi-invariance of m_X, $0 = m_X(\sigma D \cap (X \setminus Y)) = \xi_{X\setminus Y}(\xi_{\sigma D}) = \xi_{X\setminus Y}[(\xi_D)_{\sigma^{-1}}]$ for locally almost all $\sigma \in \Omega$. Now $(\xi_D)_{\sigma^{-1}}$ depends continuously on σ, so that $\xi_{X\setminus Y}[(\xi_D)_{\sigma^{-1}}] = 0$ for all $\sigma \in d\Omega$. In other words $\sigma \in d\Omega$ implies that $\sigma D \subseteq Y$ l.a.e., so that $\xi_D \in \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}$ and $D \subseteq Z$ a.e. Since the procedure is reversible, $D \subseteq Z$ a.e. implies that $D \subseteq A$ a.e., and the case $p = 1$ is completed.

Next, let $p \in (1, \infty)$. By Theorem 3.14, $\{k \in L_X^p : L_\Omega * k \subseteq L_Y^p\} = \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}^p$, which in turn is $\{k \in L_X^p : |k|^p \in \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}\}$. However, $\bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y} = L_A$ by the first part of the proof, so the collection is none other than L_A^p. For $p = \infty$, we obtain $\{k \in L^\infty(X) : L_\Omega * k \subseteq L_Y^\infty\} = \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}^\infty = \{k \in L^\infty(X) : k\xi_D \in \bigcap_{\sigma\in d\Omega} L_{\sigma^{-1}Y}^p$ for every compact $D \subseteq X\} = \{k \in L^\infty(X) : k\xi_D \in L_A$ for every compact $D \subseteq X\} = L_A^\infty$.

4. **The orbit topology and its consequences.** From now on Γ is a group of homeomorphisms of X on which there is a quasi-invariant measure m_X. We borrow the next two theorems from [4]. After that we shall assume *throughout the rest of the paper that Γ and X satisfy any and hence all the conditions of Theorem* 4.1.

4.1. Theorem. *The following three conditions are equivalent*:

(a) *If $D \subseteq X$ is compact and $m_X(D) = 0$, and if $a \in X$, then $\sigma a \notin D$ for locally almost every $\sigma \in \Gamma$.*

(b) *If* $\Phi \subseteq \Gamma$ *is a Borel set with positive measure, then for all* $a \in X$, Φa *has positive outer measure.*

(c) *Let* Γ_0 *be an open* σ*-compact subgroup of* Γ. *Then for every* $a \in X$, $\Gamma_0 a$ *is measurable and* $m_X(\Gamma_0 a) > 0$.

(We note that such Γ_0 exists in every case: every compact neighborhood of 1 generates an open σ-compact group.) The proof of this theorem is contained in 5.6 and 5.7 of [4].

The conditions of Theorem 4.1 are satisfied if X is a factor space of Γ and the action of Γ on X is the natural one [4, remark made immediately before 5.8].

4.2. Theorem. *Let* $a \in X$. *For* $\sigma \in \Gamma$ *let* $\pi_a(\sigma) = \sigma a$. *Then* π_a *is a continuous open map of* Γ *onto* Γa. *Further,* Γa *is the intersection of a closed and open set in* X, *and therefore is measurable* (*see* 5.10 *of* [4]).

Next we let $\mathscr{L}^1(X)$ be the collection of *functions* on X which are integrable (L_X still denotes the space of all classes of integrable functions that are a.e. equal), and similarly for $\mathscr{L}^\infty$. For $f \in \mathscr{L}^1(\Gamma)$ and $k \in \mathscr{L}^\infty(X)$ we put

$$(f \circ k)(x) = \int_\Gamma f(\sigma) k(\sigma^{-1}x)\, d\sigma$$

for all $x \in X$ for which the integral exists. Manifestly, if $f_1, f_2 \in \mathscr{L}^1(\Gamma)$ and $f_1 = f_2$ a.e., then $f_1 \circ k = f_2 \circ k$.

4.3. Theorem. *The following condition is equivalent to each of those stated in Theorem* 4.1:

If $f \in \mathscr{L}^1(\Gamma)$ *and* $k_1, k_2 \in \mathscr{L}^\infty(X)$ *and if* $k_1 = k_2$ *l.a.e., then* $f \circ k_1$ *and* $f \circ k_2$ *are defined everywhere on* X, *and* $f \circ k_1 = f \circ k_2$.

Proof. First we show that the above condition implies (a) of Theorem 4.1. Let D be compact in X with $m_X(D) = 0$, and let $a \in X$. By our assumption,

$$0 = (f \circ \xi_D)(a) = \int_\Gamma f(\sigma)\xi_D(\sigma^{-1}a)\, d\sigma \quad \text{for all } f \in \mathscr{L}^1(\Gamma).$$

But this just says that $\sigma^{-1}a \notin D$ for locally almost all $\sigma \in \Gamma$, which is (a). Now we utilize (b) of Theorem 4.1 to prove the above condition. We need only take $k \in \mathscr{L}^\infty(X)$ such that $k = 0$ l.a.e., and show that for each $f \in \mathscr{L}^1(\Gamma)$ and $a \in X$, $(f \circ k)(a) = 0$. By the contrapositive of (b), $k(\sigma^{-1}a) = 0$ l.a.e. in Γ. Thus $(f \circ k)(a) = \int_\Gamma f(\sigma)k(\sigma^{-1}a)\, d\sigma = 0$.

The importance of Theorem 4.3 is that we may consider $f \circ k$ as defined for $f \in L_\Gamma$, $k \in L_X^\infty$, rather than for $f \in \mathscr{L}^1(\Gamma)$, $k \in \mathscr{L}^\infty(X)$—providing the pair Γ and X satisfy the conditions of Theorem 4.1. We define $L_\Gamma \circ L_X^\infty$ as $\{f \circ k : f \in L_\Gamma, k \in L_X^\infty\}$.

Let Ω be a measurable subset of Γ and Y a measurable subset of X. By Theorem 3.9 we know there exists a measurable subset Z of X such that $L_\Omega * L_Y = L_Z$. In order to give an explicit form to a Z for which $L_\Omega * L_Y = L_Z$, we make use of two entities, one a topology on X which is customarily different from the given topology, and the second a set T in X which is directly related to the convolution and which we will show is locally the same as Z. Hereafter we designate the original topology on X by $\mathscr{T}$. We begin by discussing the new topology on X, which we denote by $\mathscr{O}$. It is described in terms of the orbits Γx of the elements x in X.

4.4. DEFINITION. A basis for the topology $\mathscr{O}$ consists of sets of the form $\{\Phi x : \Phi \text{ open in } \Gamma, x \in X\}$. We call this topology the *orbit topology* (*on* X).

The fact that the collection just described forms a basis for a bona fide topology on X is immediate. $\mathscr{O}$ is a natural analog of $\mathscr{T}$, in the sense that $\mathscr{T}$ is generated by the sets of the form $\{\sigma U : \sigma \in \Gamma, U \text{ open in } X\}$. However, the two topologies need not be identical. For an example, let $\boldsymbol{R}$ be the additive group of reals with the usual topology, and let $\Gamma = \boldsymbol{R}$ and $X = \boldsymbol{R} \cup \{\infty\}$ the one-point compactification of $\boldsymbol{R}$. Let δ_∞ be the point mass at ∞ and let m_X be defined by

$$m_X(Y) = m(R \cap Y) + \delta_\infty(Y), \quad \text{for any Borel set } Y \text{ in } X.$$

Define the action of Γ on X by

$(\sigma, x) \to x+\sigma$, $x \in \boldsymbol{R}$, $\sigma \in \Gamma$,

$(\sigma, \infty) \to \infty$, $\sigma \in \Gamma$.

This system satisfies the conditions mentioned in Theorem 4.1. The set $\{\infty\}$ is not open in $\mathscr{T}$, while $\{\infty\} = \Gamma\{\infty\}$ is both closed and open in $\mathscr{O}$.

Let us nail down a few of the properties of $\mathscr{O}$.

4.5. LEMMA. (i) *$\mathscr{O}$ is at least as fine as $\mathscr{T}$.*

(ii) *For each $x \in X$, $\mathscr{O}$ coincides with $\mathscr{T}$ on Γx.*

Proof. (i) If U is a neighborhood of a in $\mathscr{T}$, then there is an open set $\Phi \subseteq \Gamma$ such that $\Phi a \subseteq U$, since the map $(\sigma, x) \to \sigma x$ is jointly continuous. But Φa is a $\mathscr{O}$-neighborhood of a. To prove (ii) let Φ be open in Γ, and $a \in X$. By Theorem 4.2, the map $\pi_a : \Gamma \to \Gamma a$ is $\mathscr{T}$-open, so we are done.

Lemma 4.5 yields two characterizations of $\mathscr{O}$. We assume henceforth that Γ_0 is an open sigma-compact subgroup of Γ.

(i) $U \subseteq X$ is $\mathscr{O}$-open if and only if for each $x \in X$, the set $\{\sigma \in \Gamma : \sigma x \in U\}$ is open in Γ.

(ii) $U \subseteq X$ is $\mathscr{O}$-open if and only if for each $x \in X$, $U \cap \Gamma_0 x$ is relatively open in $\Gamma_0 x$.

We push on with the characteristics of $\mathscr{O}$.

4.6. LEMMA. (i) *Let $U \subseteq X$ be $\mathscr{T}$-open and $a \in X$. Then either $U \cap \Gamma_0 a = \varnothing$ or $m_X(U \cap \Gamma_0 a) > 0$.*

(ii) *Every set with finite outer measure in X intersects only countably many Γ_0-orbits.*

(iii) *Every $\mathscr{T}$-compact set is a union of countably many $\mathscr{O}$-compact sets.*

Proof. (i) Assume that $U \cap \Gamma_0 a \neq \varnothing$. Since Γ_0 is sigma-compact and $(\sigma, x) \to \sigma x$ is continuous, $\Gamma_0 a$ is sigma-compact; thus $U \cap \Gamma_0 a$ is measurable. There is an open set $\Phi \in \Gamma_0$ such that $\Phi a \subseteq U$. Thus $\Phi a \subseteq (U \cap \Gamma_0 a)$. Using Theorem 4.1(c), we have $0 < m_X(\Phi a) \leqq m_X(U \cap \Gamma_0 a)$.

To prove (ii), let $Y \subseteq X$ be of finite outer measure. Since X is locally compact there is an open set U such that $U \supseteq Y$ and $m_X(U) < \infty$. By (i), $m_X(U \cap \Gamma_0 x) > 0$ for every $x \in X$ such that $U \cap \Gamma_0 x \neq \varnothing$. Then there can be only countably many orbits $\Gamma_0 x$ with $U \cap \Gamma_0 x \neq \varnothing$, and (ii) is proved.

For (iii), let $D \subseteq X$ be $\mathscr{T}$-compact. By (ii) there exists a sequence (x_n) in X such that $D = \bigcup_{n=1}^{\infty} (D \cap \Gamma_0 x_n)$. But Γ_0 is by definition sigma-compact in Γ, whence $\Gamma_0 x_n$ is sigma-compact in X, so that $D \cap \Gamma_0 x_n$ is sigma-compact in $(X, \mathscr{T})$. Inasmuch as the original and $\mathscr{O}$-topology coincide on each Γx_n, (iii) is also proved.

By Lemma 4.6(iii), $\mathscr{T}$ and $\mathscr{O}$ have the same sigma-compact sets so the notion of l.a.e. is the same with respect to both topologies.

In the next two propositions we show that $\mathscr{O}$ is a legitimate topology to work with, with respect to Γ and m_X.

4.7. LEMMA. *Every $\mathscr{O}$-open set U is m_X-measurable, and if $U \neq \varnothing$, then $m_X(U) > 0$.*

Proof. Let U be $\mathscr{O}$-open in X. By 11.31 of [5] we only need to show that $U \cap V$ is measurable for every $V \subseteq X$ that is open in the $\mathscr{T}$-topology and has finite measure. Since V is automatically $\mathscr{O}$-open, we may assume that $U = U \cap V$. At least we know that U has finite outer measure. Now we show it is measurable. By Lemma 4.6(ii), there is a sequence $x_1, x_2, \ldots$ such that $U = \bigcup_{n=1}^{\infty} (U \cap \Gamma_0 x_n)$. Since U is $\mathscr{O}$-open and $\mathscr{T}$ and $\mathscr{O}$ coincide on $\Gamma_0 x$, for each n there is an $\mathscr{T}$-open $W_n \subseteq X$ such that $U \cap \Gamma_0 x_n = W_n \cap \Gamma_0 x_n$. Then $U = \bigcup_{n=1}^{\infty} (W_n \cap \Gamma_0 x_n)$, which is measurable. To show that if $U \neq \varnothing$ then $m_X(U) > 0$, we let $x \in U$ and find a $\mathscr{T}$-open V in X such that $x \in V \cap \Gamma_0 x = U \cap \Gamma_0 x$. By (i) of Lemma 4.6, $m_X(U) \geqq m_X(V \cap \Gamma_0 x) > 0$.

We remark that Lemma 4.7 says that if Φ is open in Γ and $x \in X$, then $m_X(\Phi x) > 0$.

4.8. THEOREM. *$(X, \mathscr{O})$ is locally compact, and Γ acts as a transformation group on $(X, \mathscr{O})$. Furthermore, m_X is a quasi-invariant Radon measure on $(X, \mathscr{O})$.*

Proof. Since each Γx is, in the $\mathscr{T}$-topology, the intersection of an open and a closed subset of X by Theorem 4.2, it is locally compact. But Γx is $\mathscr{O}$-open, so $\mathscr{O}$ is a locally compact topology for X. To show that Γ acts as a transformation group on $(X, \mathscr{O})$, we note that $\sigma \in \Gamma$ implies σ is an $\mathscr{O}$-homeomorphism. Now we show that $(\sigma, x) \to \sigma x$ is $\mathscr{O}$-jointly continuous. Let (σ, a) be fixed in $\Gamma \times X$ and U an $\mathscr{O}$-neighborhood of σa. Straight from the definition of $\mathscr{O}$, there is a neighborhood Φ of 1 in Γ such that $\sigma(\Phi\Phi)a \subseteq U$. Then $\sigma\Phi$, Φa are neighborhoods of σ, a respectively, and $(\sigma\Phi)(\Phi a) \subseteq U$.

Finally we prove that m_X is a Radon measure on $(X, \mathscr{O})$. We denote by m^* and m_* the outer and inner measure respectively of m_X. Let Y be any subset of X. Since every $\mathscr{T}$-open set is $\mathscr{O}$-open,

$$\begin{aligned} m^*(Y) &= \inf\{m_X(Z) : Y \subseteq Z, Z\ \mathscr{T}\text{-open}\} \\ &\geqq \inf\{m^*(Z) : Y \subseteq Z, Z\ \mathscr{O}\text{-open}\} \geqq m^*(Y). \end{aligned}$$

By Lemma 4.7, if Z is $\mathcal{O}$-open, then it is measurable, so we may write $m_X(Z)$ instead of $m^*(Z)$. Thus $m^*(Y)=\inf\{m(Z) : Y\subseteq Z, Z\ \mathcal{O}\text{-open}\}$. On the other hand, every $\mathcal{O}$-compact set in X is $\mathcal{T}$-compact, while every $\mathcal{T}$-compact set is the union of an increasing sequence of $\mathcal{O}$-compact sets. Therefore for any $Y\subseteq X$, $m_*(Y)=\sup\{m(D) : D\subseteq X \text{ and } D\ \mathcal{O}\text{-compact}\}$. The formulas for m^* and m_* yield m_X—the same as m_X with respect to the original topology— a Radon measure with respect to $\mathcal{O}$. Thus m_X is quasi-invariant with respect to $\mathcal{O}$.

4.9. LEMMA. (i) *$\mathcal{O}$ is the weakest topology on X for which $L_\Gamma \circ L_X^\infty \subseteq C(X)$.*

(ii) *$\mathcal{O}=\mathcal{T}$ if and only if for each compact $\Psi\subseteq\Gamma$ and each relatively $\mathcal{T}$-compact measurable $D\subseteq X$, the map $x\to m\{\sigma\in\Psi : \sigma^{-1}x\in D\}$ is continuous.*

Proof. Let $a\in X$ and let $f\in L_\Gamma$, $k\in L_X^\infty$. Then for each $\sigma\in\Gamma$,

$$(f\circ k)(\sigma a) = \int_\Gamma f(\tau)k(\tau^{-1}\sigma a)\,d\tau = (f_\sigma\circ k)(a).$$

But the shift in L_Γ is continuous, and $\pi_a\colon \Gamma\to\Gamma a$ is an open map, so on the orbit $\Gamma a, f\circ k$ is $\mathcal{O}$-continuous. Since a is arbitrary and Γa is $\mathcal{O}$-open, $f\circ k$ is $\mathcal{O}$-continuous and for the topology $\mathcal{O}$ on X, $L_\Gamma\circ L_X^\infty\subseteq C(X)$. Conversely, let U be an $\mathcal{O}$-neighborhood of $a\in X$. We shall find $f\in L_\Gamma$ and $k\in L_X^\infty$ such that $f\circ k(a)\neq 0$ and $f\circ k=0$ on $X\backslash U$; this proves that $L_\Gamma\circ L_X^\infty\subseteq C(X)$ can not hold for any topology that is strictly weaker than $\mathcal{O}$. Since Γ acts as a transformation group on $(X,\mathcal{O})$ there is a neighborhood Φ of $1\in\Gamma$ such that $m(\Phi)<\infty$ and $\Phi a\subseteq U$. Then there is an $\mathcal{O}$-open, hence measurable, set $Z\subseteq X$ such that $a\in Z$ and $\Phi Z\subseteq U$. Then $\xi_\Phi\in L_\Gamma$ and $\xi_Z\in L_X^\infty$. Clearly $\xi_\Phi\circ\xi_Z=0$ on $X\backslash U$, and $\xi_\Phi\circ\xi_Z(a)=m\{\sigma\in\Phi : \sigma^{-1}a\in Z\}>0$, $\{\sigma\in\Phi : \sigma^{-1}a\in Z\}$ being an open neighborhood of 1 in Γ.

Now we prove (ii). We note that if $\Psi\subseteq\Gamma$ is compact and $D\subseteq X$ is relatively compact, then $\xi_\Psi\circ\xi_D(x)=m\{\sigma\in\Psi : \sigma^{-1}x\in D\}$, so that the assumption $L_\Gamma\circ L_X^\infty\subseteq C(X)$ implies the condition. To prove the converse it suffices to consider any compact set $\Psi\subseteq\Gamma$ and any measurable set $Y\subseteq X$ and show that $\xi_\Psi\circ\xi_Y$ is continous at any given $a\in X$. To that end, we note that $\Psi^{-1}a$ is compact in X, so that there is a D, compact in X, whose interior contains $\Psi^{-1}a$. Let W be a neighborhood of a such that $\Psi^{-1}W\subseteq D$, and let $Q=D\cap Y$. If $y\in W$, then

$$\begin{aligned}\xi_\Psi\circ\xi_Q(y) &= m\{\sigma\in\Psi : \sigma^{-1}y\in Q\} = m\{\sigma\in\Psi : \sigma^{-1}y\in Q\cap\Psi^{-1}W\}\\ &= m\{\sigma\in\Psi : \sigma^{-1}y\in Y\cap\Psi^{-1}W\} = m\{\sigma\in\Psi : \sigma^{-1}y\in Y\} = (\xi_\Psi\circ\xi_Y)(y).\end{aligned}$$

Consequently, on W, $\xi_\Psi\circ\xi_Y=\xi_\Psi\circ\xi_Q$. Since Q is relatively compact and Ψ is compact, the hypothesis that $x\to\{\sigma\in\Psi : \sigma^{-1}x\in Q\}$ be continuous merely says that $\xi_\Psi\circ\xi_Q$ is continuous on X, and hence on W. Therefore, $\xi_\Psi\circ\xi_Y$ is continuous on W, and hence at a. Since a was arbitrary, $\xi_\Psi\circ\xi_Y$ is continuous, as desired.

In the proofs of the theorems which we are leading up to, we employ the $\mathcal{O}$-continuity of functions $f\circ k$. What will especially interest us are the sets where various $f\circ k$ are nonzero, which we know to be $\mathcal{O}$-open via Lemma 4.9(i). Let $\Omega\subseteq\Gamma$ and $Y\subseteq X$ be measurable.

4.10. Definition. Let $T=\{x\in X$: there exist $\Phi\subseteq\Gamma$, $D\subseteq X$ each with finite measure, such that $\Phi\subseteq\Omega$ l.a.e. and $D\subseteq Y$ l.a.e., and such that $(\xi_\Phi\circ\xi_D)(x)>0\}$.

We begin discussing T by proving a lemma connecting $\circ$ and $*$.

4.11. Lemma. (i) *$x\in T$ if and only if there exist compact sets $\Phi'\subseteq\Omega$ and $D'\subseteq X$ such that $(\xi_{\Phi'}\circ\xi_{D'})(y)>0$.*

(ii) *Let $\Phi\subseteq\Gamma$ and $D\subseteq X$ be compact. Then*

$$\xi_\Phi*\xi_D>0 \quad \textit{l.a.e. on } \{x\in X\colon (\xi_\Phi\circ\xi_D)(x)>0\},$$
$$\xi_\Phi*\xi_D=0 \quad \textit{l.a.e. on } \{x\in X\colon (\xi_\Phi\circ\xi_D)(x)=0\}.$$

Proof. (i) The "if" part is trivial. Now assume $x\in T$. Let Φ, D be as in the definition of T. By Theorem 4.3 and finite measures of Φ and D we may assume $\Phi\subseteq\Omega$ and $D\subseteq Y$, and also that Φ, D are sigma-compact. Let $\Phi_1,\Phi_2,\dots$ and $D_1,D_2,\dots$ be compact sets such that $\Phi=\bigcup_{k=1}^\infty\Phi_k$, $D=\bigcup_{n=1}^\infty D_n$. Then

$$0<(\xi_\Phi\circ\xi_D)(x)=m(\{\sigma\in\Phi:\sigma^{-1}x\in D\})=m\Big(\bigcup_{k,n}\{\sigma\in\Phi_k:\sigma^{-1}x\in D_n\}\Big).$$

It follows that for some k and n, $m(\{\sigma\in\Phi_k:\sigma^{-1}x\in D_n\})>0$. Put $\Phi'=\Phi_k$, $D'=D_n$, and we have $(\xi_{\Phi'}\circ\xi_{D'})(x)>0$.

(ii) For $f\in L_\Gamma$ and $k\in L_X^\infty$, we have (see §2) that $f*k(x)=\int_\Gamma f(\sigma)k(\sigma^{-1}x)J(\sigma^{-1},x)\,d\sigma$ for locally almost every x. It follows that for l.a.e. x, $(\xi_\Phi*\xi_D)(x)=0$ if and only if $\xi_\Phi(\sigma)\xi_D(\sigma^{-1}x)=0$ l.a.e. on Γ. This proves (ii).

4.12. Lemma. (i) *T is $\mathcal{O}$-open, and hence measurable.*

(ii) *$T\subseteq\Omega Y$.*

(iii) *$L_\Omega*L_Y\subseteq L_T$.*

Proof. Since each $f\circ k$ is $\mathcal{O}$-continuous, T is $\mathcal{O}$-open, and by Lemma 4.7, T is measurable, so we have (i) sewed up. To prove (ii), we note that if $x\notin\Omega Y$, then for any $\Phi\subseteq\Omega$ and $Z\subseteq Y$ with finite measure, $0=\int_\Gamma\xi_\Phi(\sigma)\xi_Z(\sigma^{-1}x)\,d\sigma=\xi_\Phi\circ\xi_Z(x)$, so $x\notin T$. Finally, we prove that $L_\Omega*L_Y\subseteq L_T$. Since for any compact sets $\Psi\subseteq\Omega$ and $D\subseteq Y$, $\xi_\Psi*\xi_D$ is 0 a.e. on $X\backslash T$ (see Lemma 4.11(ii)), we have $\xi_\Psi*\xi_D\in L_T$. But $L_\Omega*L_Y$ is generated by convolutions of such functions. Thus $L_\Omega*L_Y\subseteq L_T$.

Next we see how T is affected by the operators d and i (see the beginning of §3). Of course d and i depend upon the topology of X. If we consider d and i with respect to the $\mathcal{O}$-topology, we will denote them by $d_\mathcal{O}$ and $i_\mathcal{O}$. In $\mathcal{T}$ we will just use the plain d and i. Now we produce a lemma closely associated with Corollary 3.5.

4.13. Lemma. (i) *$d\Omega d_\mathcal{O}Y$ is contained in the $\mathcal{O}$-closure of T.*

(ii) *$d\Omega i_\mathcal{O}Y\subseteq T$.*

Proof. To prove (i), note that $L_\Omega*L_Y\subseteq L_T$. Hence $(d\Omega)(d_\mathcal{O}Y)\subseteq d_\mathcal{O}T$ by Corollary 3.5(ii). Further, $d_\mathcal{O}T$ lies in the $\mathcal{O}$-closure of T. Now we prove (ii). Let $\sigma\in d\Omega$ and $a\in i_\mathcal{O}Y$ and let D be a $\mathcal{O}$-compact neighborhood of a such that $m_X(D\backslash Y)=0$. In addition let Φ be a neighborhood of the identity in Γ of finite measure and such that $\Phi^{-1}a \subseteq D$. Inasmuch as $\tau \in \sigma\Phi \cap \Omega$ implies that $\tau^{-1}\sigma a \in \Phi^{-1}a \subseteq D$, we have the following inequalities:

$$\begin{aligned}(\xi_{\sigma\Phi\cap\Omega} \circ \xi_{D\cap Y})(\sigma a) &= \int_\Gamma \xi_{\sigma\Phi\cap\Omega}(\tau)\xi_{D\cap Y}(\tau^{-1}\sigma a)\, d\tau \qquad \text{(by Theorem 4.3)} \\ &= \int_\Gamma \xi_{\sigma\Phi\cap\Omega}(\tau)\xi_{D}(\tau^{-1}\sigma a)\, d\tau \\ &\geqq \int_\Gamma \xi_{\sigma\Phi\cap\Omega}(\tau)\, d\tau \\ &> 0 \qquad (\text{since } \sigma \in d\Omega)\end{aligned}$$

so that $\sigma a \in T$.

4.14. THEOREM. $L_\Omega * L_Y = L_T$.

Proof. In the first place, $L_\Omega * L_Y \subseteq L_T$ by Lemma 4.12(iii). To prove the reverse inclusion, we refer to the measurable $Z \subseteq X$ for which $L_\Omega * L_Y = L_Z$ (Theorem 3.9). Let us prove that $T \subseteq i_{\mathcal{O}}Z$. To begin, let $a \in T$. Then there are compact sets $\Phi \subseteq \Omega$ and $D \subseteq Y$ such that $(\xi_\Phi \circ \xi_D)(a) > 0$. If $U = \{x \in X : (\xi_\Phi \circ \xi_D)(x) > 0\}$, then U is an $\mathcal{O}$-neighborhood of a, and since $\xi_\Phi \circ \xi_D > 0$ on U, we must have $U \subseteq Z$ l.a.e. Thus $a \in i_{\mathcal{O}}Z$. Consequently $T \subseteq i_{\mathcal{O}}Z \subseteq Z$ l.a.e., so that $L_T \subseteq L_Z = L_\Omega * L_Y$.

5. **Vanishing modules.** The central problem we tackle in this section is the relationship between the two expressions $\Omega Y \subseteq Y$ and $L_\Omega * L_Y \subseteq L_Y$. They look as though they ought to be related; perhaps they are even equivalent. However, even at the outset trouble looms, because $\Omega Y \subseteq Y$ is set-theoretic and $L_\Omega * L_Y \subseteq L_Y$ is measure-theoretic. Nevertheless, let us work on the problem and see what we can harvest.

5.1. THEOREM. *If* $\Omega Y \subseteq Y$, *then* $L_\Omega * L_Y \subseteq L_Y$.

Proof. Straightforward from the convolution formula (see §2).

That was simple. However, the converse of Theorem 5.1 is by no means so simple, even if $X = \Gamma$. A conjecture arose in [10] that if $\Omega \subseteq \Gamma$ is relatively sigma-compact in Γ, and if $L_\Omega * L_\Omega \subseteq L_\Omega$, then there exists an $\Omega' \subseteq \Gamma$ such that $\Omega' = \Omega$ a.e. and $\Omega'\Omega' \subseteq \Omega'$. Finally this was proved in [6]. For Ω which is not relatively sigma-compact the answer is yet unknown, so far as we can determine. Consequently the extension from subalgebras of L_Γ to submodules of L_X can be expected to bring great difficulty.

A subalgebra L_Ω of L_Γ with the property that $L_\Omega * L_\Omega \subseteq L_\Omega$ has been christened a vanishing algebra [10], since it vanishes outside Ω. A subspace L_Y of L_X with the property that for a specific L_Ω in L_Γ one has $L_\Omega * L_Y \subseteq L_Y$ might be called a vanishing submodule of L_X. In our proposition we refer to symbolic notation for clarity and simplicity.

5.2. THEOREM. *Let* $\Omega \subseteq \Gamma$ *and* $Y \subseteq X$ *be measurable relatively sigma-compact sets. Then there exist* $\Omega_0 \subseteq \Gamma$ *and* $Y_0 \subseteq X$ *such that* $\Omega_0 = \Omega$ *a.e., and* $Y_0 = Y$ *a.e., and such that* $\Omega_0 Y_0 = T$.

Proof. For the present we assume that Ω and Y are relatively compact. For each $x \in X$, let $Y_x = \{\sigma \in \Gamma : \sigma^{-1}x \in Y\}$. If Φ is any compact neighborhood of $1 \in \Gamma$, define $f_{\Phi,Y}$ and $g_{\Phi,\Omega}$ by

$$f_{\Phi,Y}(x) = m(Y_x \cap \Phi)/m(\Phi), \qquad x \in X,$$
$$g_{\Phi,\Omega}(\sigma) = m(\sigma^{-1}\Omega \cap \Phi)/m(\Phi), \qquad \sigma \in \Gamma.$$

First we note that $f_{\Phi,Y}$ and $g_{\Phi,\Omega}$ are measurable. Given any measurable set $Z \subseteq X$, we have

$$\begin{aligned} \left| \int_Z [\xi_Y(x) - f_{\Phi,Y}(x)]\, dx \right| &= \left| \frac{1}{m(\Phi)} \int_Z \int_\Phi [\xi_Y(x) - \xi_{Y_x}(\sigma)]\, d\sigma\, dx \right| \\ &\leqq \frac{1}{m(\Phi)} \int_\Phi d\sigma \int_Z |\xi_Y(x) - \xi_{\sigma Y}(x)|\, dx \\ &= \frac{1}{m(\Phi)} \int_\Phi m_X(Z \cap (Y \Delta \sigma Y))\, d\sigma \\ &\leqq \sup_{\sigma \in \Phi} m_X(Y \Delta \sigma Y), \end{aligned}$$

where Δ is used to denote the symmetric difference of sets. Thus $\|\xi_Y - f_{\Phi,Y}\|_1 \leqq 2 \sup_{\sigma \in \Phi} m_X(Y \Delta \sigma Y)$. Similarly, $\|\xi_\Omega - g_{\Phi,\Omega}\|_1 \leqq 2 \sup_{\sigma \in \Phi} m(\Omega \Delta \Omega \sigma^{-1})$. Since Ω and Y are relatively compact, the functions $\sigma \to m_X(Y \Delta \sigma Y)$ and $\sigma \to m(\Omega \Delta \Omega \sigma^{-1})$ are continuous at 1 by Corollary 3.6 of [4]. Hence, for every $\varepsilon > 0$ there exists a neighborhood Φ of 1 such that $\|\xi_Y - f_{\Phi,Y}\|_1 < \varepsilon$ and $\|\xi_\Omega - g_{\Phi,\Omega}\|_1 < \varepsilon$.

Now we drop the condition that Ω and Y be relatively compact. By the hypotheses, there exist increasing sequences $(\Omega_n)_{n \in N}$ and $(Y_n)_{n \in N}$ of relatively compact sets in Γ and X respectively such that $\Omega = \bigcup_n \Omega_n$ and $Y = \bigcup_n Y_n$. For each n we may choose a compact neighborhood Φ_n of 1 such that $\|\xi_{\Omega_n} - g_{\Phi_n,\Omega_n}\|_1 \leqq n^{-2}$ and $\|\xi_{Y_n} - f_{\Phi_n,Y_n}\|_1 \leqq n^{-2}$. Since $0 \leqq 1 - f_{\Phi_n,Y} \leqq 1 - f_{\Phi_n,Y_n}$, we obtain

$$\|[1 - f_{\Phi_n,Y}]\xi_{Y_n}\|_1 \leqq \|[1 - f_{\Phi_n,Y_n}]\xi_{Y_n}\|_1 \leqq \|\xi_{Y_n} - f_{\Phi_n,Y_n}\|_1 \leqq n^{-2}.$$

It follows that $\lim_n \{[1 - f_{\Phi_n,Y}]\xi_{Y_n}\} = 0$ a.e., so that $\lim_n f_{\Phi_n,Y} = 1$ a.e. on Y. In the same way one can prove that $\lim_n g_{\Phi_n,\Omega} = 1$ a.e. on Ω.

We have set the stage for Ω_0 and Y_0. Let $\Omega_0 = \{\sigma \in \Omega : \lim_n g_{\Phi_n,\Omega}(\sigma) = 1\}$, and let $Y_0 = \{x \in Y : \lim_n f_{\Phi_n,Y} = 1\}$. Then $\Omega_0 = \Omega$ a.e. and $Y_0 = Y$ a.e. We already know from Lemma 4.12(ii) that $T \subseteq \Omega_0 Y_0$. To prove the opposite inclusion, let $\sigma \in \Omega_0$, $y \in Y_0$. There is an n such that $f_{\Phi_n,Y}(y) > \frac{1}{2}$ and $g_{\Phi_n,\Omega}(\sigma) > \frac{1}{2}$. If we let $\Phi = \Phi_n$ then Φ is a neighborhood of 1 in Γ such that $m(\sigma^{-1}\Omega \cap \Phi) > \frac{1}{2}m(\Phi)$ and $m_X(Y_y \cap \Phi) > \frac{1}{2}m(\Phi)$. Then

$$\begin{aligned} m(\Phi \cap \sigma^{-1}\Omega \cap Y_y) &= m(\Phi \cap \sigma^{-1}\Omega) + m(\Phi \cap Y_y) - m(\Phi \cap (\sigma^{-1}\Omega \cup Y_y)) \\ &> \tfrac{1}{2}m(\Phi) + \tfrac{1}{2}m(\Phi) - m(\Phi) = 0. \end{aligned}$$

Thus

$$\begin{aligned}\xi_{\sigma\Phi\cap\Omega}\circ\xi_{\Phi^{-1}y\cap Y}(\sigma y) &= m(\sigma\Phi\cap\Omega\cap(\Phi^{-1}y\cap Y)_{\sigma y})\\ &= m(\sigma\Phi\cap(\Phi^{-1}y)_{\sigma y}\cap\Omega\cap Y_{\sigma y})\\ &= m(\sigma\Phi\cap\Omega\cap Y_{\sigma y})\\ &= m(\Phi\cap\sigma^{-1}\Omega\cap Y_y)\\ &> 0,\end{aligned}$$

which means that $\sigma y\in T$, because $\xi_{\sigma\Phi\cap\Omega}\in L_\Omega$ and $\xi_{\Phi^{-1}y\cap Y}\in L_Y$.

5.3. THEOREM. *Let $\Omega\subseteq\Gamma$ and $Y\subseteq X$ be measurable relatively sigma-compact sets. Assume also that $L_\Omega * L_Y\subseteq L_Y$. Then there exist $\Omega_1\subseteq\Gamma$ and $Y_1\subseteq X$ such that $\Omega_1=\Omega$ a.e. and $Y_1=Y$ a.e., and such that $\Omega_1 Y_1\subseteq Y_1$.*

Proof. By the preceding theorem, $\Omega_0 Y_0=T$. In view of the assumption $L_\Omega * L_Y\subseteq L_Y$ and the identity $L_Y=L_{Y_0}$, Theorem 4.14 tells us that $T\subseteq Y_0$ l.a.e. Since T is $\mathcal{O}$-open, $T\subseteq i_{\mathcal{O}}Y_0$. But then $(d\Omega_0)T\subseteq T$ by Lemma 4.13(ii). Now let $\Omega_1=\Omega_0\cap d\Omega_0$ and $Y_1=Y_0\cup T$. Then $\Omega_1=\Omega$ l.a.e., $Y_1=Y$ l.a.e., and $\Omega_1 Y_1\subseteq\Omega_0 Y_0\cup(d\Omega_0)T\subseteq T\subseteq Y_1$.

We mention that Theorem 5.3 generalizes Theorem 4.3 of [6]. We do not know if we may eliminate the sigma-compactness hypothesis. Under certain circumstances—when Y is either open or closed—we can, as demonstrated in Corollaries 3.12 and 3.13.

As might be expected, if we further restrict our attention to those $\Omega\subseteq\Gamma$ and $Y\subseteq X$ for which $L_\Omega * L_Y=L_Y$, we can furnish more explicit information. For this case we give a complete solution to the converse of Theorem 5.1. First we prove the theorem for $X=\Gamma$.

5.4. THEOREM. *$L_\Omega * L_\Omega=L_\Omega$ if and only if there is an open set Ω_0 in Γ such that $\Omega_0=\Omega$ l.a.e. and such that $\Omega_0\Omega_0=\Omega_0$.*

Proof. Assume that $L_\Omega * L_\Omega=L_\Omega$. We will show that we can take Ω_0 to be T. By Theorem 4.14, $T=\Omega$ l.a.e., by Lemma 4.12(i) T is open ($\mathcal{O}=\mathcal{T}$!). In order that the definition of Ω_0 as T satisfy the conclusions of the theorem, we need only show that $TT=T$. Since

$$TT\subseteq(i_{\mathcal{O}}T)(i_{\mathcal{O}}T)=(i_{\mathcal{O}}\Omega)(i_{\mathcal{O}}T)\subseteq(d_{\mathcal{O}}\Omega)(i_{\mathcal{O}}T),$$

Lemma 4.13(ii) yields $TT\subseteq T$, whereas $T\subseteq TT$ by Lemma 4.12(ii) since $L_T=L_\Omega$.

To prove the converse, we first show that $\Omega=T$ l.a.e. By Lemma 4.12(ii) $T\subseteq\Omega_0\Omega_0=\Omega_0$, while by Lemma 4.13(ii), $\Omega_0\Omega_0\subseteq T$, since Ω_0 is open. Then Theorem 4.14 wraps it up: $L_\Omega * L_\Omega=L_T=L_\Omega$.

5.5. THEOREM. *$L_\Omega * L_Y=L_Y$ if and only if there exist $\Omega_0\subseteq\Omega$ and $\mathcal{O}$-open $Y_0\subseteq X$ such that $\Omega_0=\Omega$ l.a.e., $Y_0=Y$ l.a.e., and such that $\Omega_0 Y_0=Y_0$.*

Proof. We show that $\Omega_0=\Omega \cap d\Omega$ and $Y_0=T$ satisfy the requirements, under the assumption that $L_\Omega * L_Y=L_Y$. Clearly $\Omega_0=\Omega$ l.a.e. By virtue of Theorem 4.14, $T=Y$ l.a.e. This means that $L_{\Omega_0} * L_T=L_T$, so that by Lemma 4.12(ii) $T\subseteq\Omega_0 T$. On the other hand, $\Omega_0 T\subseteq(d\Omega)(i_{\mathcal{O}}T)=(d\Omega)(i_{\mathcal{O}}Y)\subseteq T$ by Lemma 4.13(ii). The converse can be proved virtually the same as the converse in Theorem 5.4.

Assuredly, if $L_\Gamma \circ L_X^\infty \subseteq C(X)$, then T is open in $\mathscr{T}$, so we obtain Y_0 open in that topology.

If $1 \in d\Omega$, then by Lemma 3.1, L_Ω contains an approximate identity $(u_\iota)_{\iota\in I}$ of L_Γ. For every $k \in L_Y$ we then have $k=\lim_\iota u_\iota * k \in L_\Omega * L_Y$. Thus we obtain

5.6. THEOREM. *If* $1 \in d\Omega$ *and* $L_\Omega * L_Y\subseteq L_Y$, *then* $L_\Omega * L_Y=L_Y$.

5.7. COROLLARY. *Let* $\Omega\subseteq\Gamma$ *be such that* $1\in d\Omega$ *and* L_Ω *is a subalgebra of* L_Γ. *Then there is an open* $\Omega_0\subseteq T$ *such that* $\Omega_0=\Omega$ *l.a.e. and* $\Omega_0\Omega_0=\Omega_0$.

It is worth noticing that Γ may well contain open subsets Ω with $\Omega\Omega=\Omega$ while $1\notin d\Omega$. For an example, let Γ be the additive group of the reals with the discrete topology, and $\Omega=\{\sigma\in\Gamma : \sigma>0\}$.

REFERENCES

1. N. Bourbaki, *Intégration*. Chapitre 6: *Intégration vectorielle*, Actualités Sci. Indust., no. 1281, Hermann, Paris, 1959. MR **23** #A2033.

2. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. **26** (1959), 199–205. MR **21** #3729.

3. S. L. Gulick, T.-S. Liu and A. C. M. van Rooij, *Group algebra modules*. I, Canad. J. Math. **19** (1967), 133–150. MR **36** #5712.

4. ———, *Group algebra modules*. II, Canad. J. Math. **19** (1967), 151–173. MR **36** #5713.

5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. 1: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York and Springer-Verlag, Berlin and New York, 1963. MR **28** #158.

6. T.-S. Liu, *Invariant subspaces of some function spaces*, Quart. J. Math. Oxford Ser. (2) **14** (1963), 231–239. MR **27** #1561.

7. T.-S. Liu and A. C. M. van Rooij, *Transformation groups and absolutely continuous measures*. II, Nederl. Akad. Wetensch. Proc. Ser. A **73** = Indag. Math. **32** (1970), 57–61.

8. ———, *Sums and intersections of normed linear spaces*, Math. Nachr. **42** (1969), 29–42.

9. W. Rudin, *Measure algebras on abelian groups*, Bull. Amer. Math. Soc. **65** (1959), 227–247. MR **21** #7404.

10. A. B. Simon, *Vanishing algebras*, Trans. Amer. Math. Soc. **92** (1959), 154–167. MR **23** #A1240.

UNIVERSITY OF MARYLAND,
COLLEGE PARK, MARYLAND 20742
UNIVERSITY OF MASSACHUSETTS,
AMHERST, MASSACHUSETTS 01002
CATHOLIC UNIVERSITY,
NIJMEGEN, THE NETHERLANDS